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On the Algebraic Varieties Containing a Curve in Projective Space (代数幾何学への可換環論の 応用)

AUTHOR(S):

GRECO, SILVIO; VALABREGA, PAOLO

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On the algebraic varieties containing a curve in projective space

Silvio Greco

Paolo Valabrega

Introduction.

Let Z be a projective curve in P^n over an algebraically closed field k ; we want to investigate the linear system of all hypersurfaces of P^n through Z with degree $t \gg 0$, in order to see whether some member of such a system is smooth everywhere or has at least "good" tangent cones at the singularities of Z . Moreover we want to investigate the finite intersections of general hypersurfaces through Z , to see whether there are r -dimensional algebraic varieties containing Z and smooth (as far as it is possible). When Z is smooth or has just plane singularities (embedding dimension 2), then there is a surface, global complete intersection in P^n of hypersurfaces through Z , which contains Z and is smooth everywhere; more generally, when Z has at most points with embedding dimension $r \leq n-1$, then there is a smooth r -dimensional variety through Z , global complete intersection in P^n .

When Z has singular points of embedding dimension n , we look for hypersurfaces through Z smooth everywhere else and having at these points ordinary singularities (i.e. with tangent cone projectively smooth). There are examples of "bad" singular points of Z for which such a hypersurface cannot exist. So we introduce the concept of "quasi ordinary" singular point: when $P \in Z \subset P^n$ has embedding dimension n it is quasi ordinary if the projectivized tangent cone to Z at P , as a zero dimensional subvariety of P^{n-1} , is contained as a subscheme in some smooth variety. If all points of Z with em

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embedding dimension n have such a property, then we are able to produce a hypersurface F through Z smooth everywhere except at these bad points, where the tangent cones to F are projectively smooth. In particular a curve Z in P^3 just with quasi ordinary singularities lies on a surface which is smooth elsewhere and has ordinary tangent cones at these points.

For embedding dimension $r \leq n-1$, the existence of a smooth r -fold through Z is also proved in [A-K], with different techniques (which seem less elementary than ours).

One of the main tools of our work consists of a Bertini theorem (on the variable singular point), valid for any algebraically closed field useful to deal with the behaviour of the hypersurfaces at the points of $P^n - Z$ and at the points of embedding dimension n of Z . It includes the analogous Bertini theorem of [A-K], but our proof seems quicker and more elementary.

A Bertini type method works also to deal with the behaviour of the generic hypersurface through Z at the simple points of Z .

0. Conventions and terminology.

0.1 We work over an algebraically closed field k and "algebraic variety" means "scheme of finite type" over k .

By point we mean "closed point" and we often identify an algebraic variety with the ringed space of its closed points.

If Z, Y are closed subvarieties of X we write $Z \subset Y$ to mean that Z is a subvariety of Y .

We write P^n for projective space over k .

0.2 We consider only Cartier divisors. If a divisor is effective, we identify it with the closed subvariety corresponding to it.

0.3 The intersection of two closed subvarieties is always in the algebraic sense; that is if Y_1, Y_2 are the subvarieties of X given by the sheaves of ideals I_1, I_2 , then $Y_1 \cap Y_2$ is the closed subvariety given by $I_1 + I_2$.

0.4 If $Y \subset X$ is a closed subvariety and D is a divisor of X not containing any component of Y , then $D \cap Y$ is a divisor on Y and is called "the divisor cut out by D on Y ".

If S is a linear system on X , the restriction of S to Y is the set S' of divisors on Y which can be expressed in the form $D \cap Y$ where $D \in S$. It is clear that S' is a linear system. More precisely if $j : Y \rightarrow X$ is the embedding and if S corresponds to the vector space $V \subset H^0(X, L)$, L being an invertible sheaf, then S' corresponds to the vector space $V' = \varphi(V)$, where $\varphi : H^0(X, L) \rightarrow H^0(Y, j^*L)$ is the canonical map.

0.5 If S is a linear system on X and $D_0, \dots, D_r \in S$, $\lambda = (\lambda_0, \dots, \lambda_r) \in P^r$, we write D_λ for the linear combination $\lambda_0 D_0 + \dots + \lambda_r D_r$.

The "+" sign is always used for linear combinations (and not for sum of divisors) unless explicitly stated.

0.6 Let S be a finite dimensional linear system over X and let D_0, \dots, D_r be a fixed base of S . We say that the generic element of S verifies a given property P if the set $\{\lambda \in P^r / D_\lambda \text{ has } P\}$ contains a non empty open set. This is clearly independent on the choice of the base and of the coordinates in P^r .

0.7 When considering local equations of the divisors of a linear system S (at a point or on an open set) we always take the equations in a coherent way. That is to say : if S corresponds to $V \subset H^0(X, L)$, the local equations in $U \subset X$, where $L|_U \cong \mathcal{O}_{X/U}$ (or in $x \in X$) are given by one, and the same for all $D \in S$, fixed isomorphism $L|_U \cong \mathcal{O}_{X/U}$ (resp. $L_x \cong \mathcal{O}_x$).

0.8 The base locus of a linear system S on X is the set of points of X which belong to all $D \in S$; it is a closed subset of X .

1. A Bertini theorem.

In this section we give sufficient conditions for the generic member of a linear system to be smooth outside its base locus.

1.0 Notations. Throughout this section X is a proper integral variety of dimension $d > 0$ over k , $Z \subset X$ is a closed reduced subvariety and $U = X - Z$.

S is a linear system on X , of dimension r , and D_0, \dots, D_r form a fixed base for S . For each $\lambda = (\lambda_0, \dots, \lambda_r) \in P^r$ we put : $D_\lambda = \sum_i \lambda_i D_i$ (0.5).

1.1 Definition : We say that D separates $x, y \in X$ if $x \in D, y \notin D$. We say that S separates the points of $U \subset X$ if, whenever $x, y \in U$ and $x \neq y$, there is $D \in S$ which separates x and y . Note that in this case the base locus of S is contained in $Z = X - U$ (but not conversely). We say that $E_0, \dots, E_n \in S$ separate the tangent vectors at $x \in X$ if $x \in E_0 \cap \dots \cap E_n$ and the local equations of the E_i 's at x generate the maximal ideal \mathfrak{m}_x of $\mathcal{O}_{X,x}$, i.e. they are linearly independent modulo \mathfrak{m}_x^2 (see 0.7 for conventions about local equations).

We say that S separates the tangent vectors at $x \in X$ if there are $E_0, \dots, E_n \in S$ as above (see [H], 7.3, p.152). If this happens for all $x \in U$ we say that S separates the tangent vectors on U .

We begin with a well known lemma :

1.2 Lemma : Let $f : V \rightarrow W$ be a dominant morphism of algebraic varieties. Then $\dim V \geq \dim W$.

Proof : Let W_0 be an irreducible component of W having maximal dimension and let V_0, \dots, V_n be the irreducible components of V . Then we have : $W = \overline{f(V)} = \bigcup_i \overline{f(V_i)}$, hence $W_0 = \bigcup_i (\overline{f(V_i)} \cap W_0) = \bigcup_i (\overline{f(V_i)} \cap W_0)$; therefore we may assume that $W_0 = \overline{f(V_0)}$. Therefore we may assume that both V and W are irreducible; moreover it is easy to see that they may be assumed also reduced. Now use the transcendence degrees over the base field.

1.3 Theorem : With the notation of 1.0 assume that

- (i) $U \subset \text{Reg}(X)$;
- (ii) The base locus of S is contained in Z (e.g. S separates the points of U);
- (iii) S separates the tangent vectors on U .

Then the generic element of S is nonsingular at the points of U
(i.e. for $D \in S$ generic $\text{Sing}(D) \subset Z$).

Proof (compare with [H], proof of Bertini theorem, II, 8.18) : Put :

$$B = \{(x, \lambda) \in U \times \mathbb{P}^r / x \in \text{Sing}(D_\lambda)\}.$$

Claim 1 : B is closed in $U \times \mathbb{P}^r$.

Indeed let $U = \bigcup_i U_j$ be an open affine covering such that D_i is principal on U_j , for all i, j . Clearly it is sufficient to show that $B \cap (U_j \times \mathbb{P}^r)$ is closed for all j ; so we may assume that U be affine and D_i be principal on U for all i .

Fix a closed embedding $U \subset \mathbb{A}^n$ and let $I = (G_1, \dots, G_s) \subset k[T] = k[T_1, \dots, T_n]$ be the corresponding ideal. For each $i = 0, \dots, r$ there is a hypersurface F_i such that $D_i \cap U$, in the above embedding, corresponds to the divisor cut out by F_i on U . Then for each λ the variety $D_\lambda \cap U$ is isomorphic to the subvariety of \mathbb{A}^n defined by the ideal $(G_1, \dots, G_s, \sum \lambda_i F_i)$, so that claim 1 follows easily by the Jacobian criterion.

Claim 2 : $\dim B < r$. Indeed consider on B the reduced structure and let $p : B \rightarrow U$ be the canonical morphism. For each $x \in U$ let B_x be the fiber of p at x . Since $\dim O_{X,x} = d$ for all x , it is sufficient to show that $\dim B_x = r-d-1$ for all x (see [M], (13.B)). Fix $x \in U$ and put $A = O_{X,x}$. Let \mathfrak{m} be the maximal ideal of A . Denote by $q : U \times \mathbb{P}^r \rightarrow \mathbb{P}^r$ the canonical morphism and for each i let f_i be a local equation of D_i at x . Then we have :

$$B_x \cong q(B_x) = \{\lambda \in \mathbb{P}^r / \sum \lambda_i f_i \in \mathfrak{m}^2\}.$$

Thus $B_x = \mathbb{P}(\text{Ker } \varphi)$, where φ is the linear map from k^{r+1} to A/\mathfrak{m}^2 defined by $\varphi(\lambda_0, \dots, \lambda_r) = \sum \lambda_i f_i \text{ mod. } \mathfrak{m}^2$.

Now by (ii) and (iii) it follows that φ is surjective; by (i) we have $\dim A/\mathfrak{m}^2 = d+1$, so claim 2 follows.

Now we can conclude the proof of the theorem. Let \bar{B} be the closure of B in $X \times P^r$ and let $\pi : X \times P^r \rightarrow P^r$ be the projection. Since X is proper over k we have that $\pi(\bar{B})$ is closed; and then by 1.2 we have $\dim \pi(\bar{B}) \leq \dim \bar{B} = \dim B < r$. Thus $P^r - \pi(\bar{B})$ is open and dense in P^r . On the other hand if $\lambda \in P^r - \pi(\bar{B})$ it is clear that $\text{Sing}(D_\lambda) \cap U = \emptyset$ and the conclusion follows.

1.4 Lemma : Let $X' \subset X$ be a closed irreducible subvariety and put : $U' = U \cap X'$. Let S' be the restriction of S to X' (see 0.4). If S separates the points (and the tangent vectors) on U , then S' separates the points (and the tangent vectors) on U' .

Proof : If $x, y \in U'$, $x \neq y$, then there is $D \in S$ which separates x, y ; hence $D \cap X'$ and $D' = D \cap X'$ is an element of S' which separates x and y . Assume S separates the tangent vectors on U . Let $x \in U'$ and put : $S_x = \{D \in S / x \in D\}$. Then S_x is a linear system. If $U' = \{x\}$ there is nothing to prove; so we may assume that there is $y \in U'$, $y \neq x$. Then there is $D \in S_x$, with $y \notin D$, and hence the generic element of S_x does not contain X' . Hence the tangent vectors at x , viewed as a point of U , can be separated by $E_0, \dots, E_n \in S$, with $E_i \cap X' \neq X'$. Therefore $E_0 \cap X', \dots, E_n \cap X'$ are elements of S' , which separate the tangent vectors at x , viewed as a point of U' .

1.5 Corollary : Let $Y \subset X$ be a closed subvariety and let $V = Y \cap U$. Assume that S separates points and tangent vectors on U and that $V \subset \text{Reg}(Y)$. Then if $D \in S$ is generic we have : $\text{Sing}(D \cap Y) \subset Y - V$.

Proof : Let X' be an irreducible component of Y and put : $U' = U \cap X'$. Then, if S' is the restriction of S to X' , by 1.3 and 1.4 we have that the generic element of S' is smooth on U' . We may assume that

$D'_i = D_i \cap X'$ for $i = 1, \dots, s$ form a basis of S' and that $D_i \cap X'$ for $i > s$. Let $T' \subset P^r$ be closed such that if $(\lambda_0, \dots, \lambda_r) \notin T'$ then $D'_\lambda = \sum \lambda_i D'_i$ is smooth on U' . Let $T = \{(\lambda_0, \dots, \lambda_r) \in P^r / (\lambda_0, \dots, \lambda_s) \in T'\}$.

Then T is a closed subset of P^r , different from P^r , and it is easy to see that if $\lambda \notin T$ then $D_\lambda \cap X'$ is smooth at the points of U' . The conclusion follows by the irreducibility of P^r .

1.6 Corollary : Let the notations and the assumptions be as in 1.3 and assume that S separates the points of U . Then for any $n > 0$ there are $E_1, \dots, E_n \in S$ such that

- (a) $E_1 \cap \dots \cap E_n \cap U$ is smooth;
- (b) $\dim E_1 \cap \dots \cap E_n = d-n$ ($E_1 \cap \dots \cap E_n = \emptyset$ if $d-n < 0$);
- (c) E_1 is generic in S , E_2 can be any D_λ where λ varies in a non-empty open subset of P^r (depending on E_1) and so on.

Proof : it follows from 1.5 by induction on n .

1.7 Definition : If E_1, \dots, E_n verify (ii) and (iii) of 1.6 we say that they are generically independent (This clarifies the notion used in [A-K], th. 1).

Now we want to apply the above to the case when X is projective.

1.8 Lemma : Assume $X \subset P^n$ and for $t > 0$ let $S^{(t)}$ be the linear system cut out on X by the hypersurfaces of degree t which contain Z . Assume that for some t the base locus of $S^{(t)}$ is Z . Then for all $t' > t$ $S^{(t')}$ separates the points and the tangent vectors on U .

Proof : Enough to prove the claim for $t' = t+1$. Let $x, y \in U$, $x \neq y$. Then there is $D \in S^{(t)}$ with $y \notin D$. Thus if H is a hyperplane section containing x but not y , $D+H$ (sum of divisors) is an element of $S^{(t+1)}$ which separates x and y .

Let now $x \in U$ and let H_1, \dots, H_ℓ be hyperplane sections which separate the tangent vectors at x . Let $D \in S$ with $x \in D$. Then $D+H_1, \dots, D+H_\ell$ (sum of divisors) are element of $S^{(t+1)}$ which separate the tangent vectors at x .

1.9 Corollary : For all $t \gg 0, S^{(t)}$ separates points and tangent vectors on U (hence 1.3 is applicable).

Proof : There is t_0 such that Z is set theoretic intersection of hypersurfaces of degree t_0 . Then any $t > t_0$ works by 1.8.

1.10 Corollary : Let $Y \subset P^n$ and $W \subset Y$ be closed subvarieties. Assume that W is (set theoretically) the intersection of a family of hypersurfaces of degree t_0 and that $Y-W$ is smooth. Then, if G_1, \dots, G_n are generic independent hypersurfaces of degree $t > t_0$ (see 1.7), then $G_1 \cap \dots \cap G_n \cap (Y-W)$ is smooth (possibly empty).

Proof : Apply 1.8 and 1.9 with $X = P^n$, $U = P^n - W$.

1.11 Corollary : 1.10 remains true also if we delete the assumption " k algebraically closed", provided k is infinite.

Proof : It follows easily from 1.10, by extension of the ground field to the algebraic closure.

1.12 Corollary : Let $W \subset P_k^n$ (k infinite, not necessarily algebraically closed) and let $\dim W = d$. Then for all e with $d < e \leq m$ there

is a complete intersection Y ^(of dimension e) containing W and such that $Y-W$ is smooth.

1.13 Remarks : (i) In characteristic 0 theorem 1.3 holds without assumption (iii): this is in fact the classical "second theorem of Bertini" (on the variable singular point : see [Z] , section 5).
(ii) In positive characteristic theorem 1.3 is false without assumption (iii): see for instance the example given at the end of [Z] .
(iii) Other formulations of the second theorem of Bertini in characteristic $p > 0$ can be found in the literature : see e.g. [A] . For related results see also [F] , section 5.

(iv) It is not true that in 1.6 it is enough to take E_0, \dots, E_n generic, linearly independent and such that $\dim E_1 \cap \dots \cap E_n = d-n$. For example let $U = X = P^2$ and let S be the linear system of all conics.

Assume there is an open non empty $W \subset P^5$ such that, whenever $\lambda, \mu \in W$, D_λ and D_μ are non singular and also $D_\lambda \cap D_\mu$ is nonsingular (i.e. D_λ and D_μ are not tangent). Fix $C = D_\lambda$, $\lambda \in W$, and fix $P \in C$. Let C' be a nonsingular conic, different from C , tangent to C at P . Then the pencil generated by C and C' corresponds to a line in P^5 which intersects W . Thus there is $\mu \neq \lambda$ such that $\mu \in W$ and D_μ is tangent to D_λ at P , a contradiction.

(v) The following conjecture regarding 1.7 (ii) seems reasonable :
" there is a nonempty open subset $W \subset P^r \times \dots \times P^r$ (n times) such that $D_{\lambda_1} \cap \dots \cap D_{\lambda_n} \cap U$ is nonsingular for all $(\lambda_1, \dots, \lambda_n) \in W$ ".

(vi) The results of [A-K] and [F] and of the present paragraph seem to support the following conjecture : Let X be proper over k and irreducible of dimension $d > 0$. Let $Z \subset X$ be closed of codimension at least 2. Let S be a linear system on X which separates

points and tangent vectors on $X-Z$. Then the generic element of S is irreducible.

(vii) Note that in the above conjecture the assumption on the co-dimension of Z is necessary. Indeed let X be a nonsingular quadric in P^3 and let L_1, L_2 be two skew lines on X ; then put $Z = L_1 \cap L_2$. Then $S = S^{(2)}$ (notations of 1.8) separates points and tangent vectors on $U = X-Z$, but all elements of S contain the two skew lines, so being not irreducible.

(viii) Lemma 1.1 is false for general schemes. For example take $V = \text{Spec}(A_p)$, $W = \text{Spec}(A)$, A non local.

ix) The theorem of Bertini for $S =$ linear system of all hypersurfaces of degree $t > 0$ in P^N is really stronger : the set of all nonsingular elements of S is open and $\neq \emptyset$, as it can be easily seen remarking that the set B of the proof of theorem 1.3 is closed in the projective variety $P^N \times P^r$, hence $\phi(B)$ is closed $\neq P^r$.

2. Behaviour of divisors at the points of low embedding dimension of a base curve.

In this section we keep the notations of 1.0 and we assume further that Z is a curve contained in every element of S .

Our aim is to study the behaviour of the generic element of S at the points of Z . In this section we begin with the points of low embedding dimension (see theorem 2.3).

If $x \in X$ and $D, D' \in S$ we say that D and D' are non tangent at x if $D \cap D'$ is non singular at x and $D \neq D'$. This is equivalent to say that the leading forms of their local equations at x (see 0.7) are linearly independent elements of $\text{gr}(O_{X,x})$ of degree 1.

2.1 Proposition : Let $V \subset \mathbb{Z}$ be an open subset and let $I \subset V$ be finite. Assume that:

(*) for each $x \in V - I = V'$ there are $D, D' \in S$ non tangent at x ;

and (**) for each $y \notin I$ there is $D \in S$ nonsingular at y .

Then the generic element of S is nonsingular at all points of V .

Proof : Clearly by (**) the generic element of S is nonsingular at each point of I and hence we may assume that $I = \emptyset$.

Let $W = \{(x, \lambda) \in V \times P^r / x \in \text{Sing}(D_\lambda)\}$. Then W is closed in $V \times P^r$ (same proof as in 1.2, claim 1). Moreover by the jacobian criterion the generic element of S has only finitely many singular points on V . Hence there is a non empty open set $U \subset P^r$ such that the fibers of $p : W \rightarrow P^r$ at the points of U are finite. By [EGA], IV₃, (9.7.8), we may also assume that the number of geometric connected components of the fibers over U be equal to the number of geometric connected components of the fiber over the generic point of P^r ; but, if $x \in U$ is closed and if the base field is algebraically closed, such a number is simply the cardinality of the fiber over x ([EGA], IV₂, (4.5.2)). So we may assume that the fibers over the points of U are finite and of constant cardinality s . Our aim is to show that $s = 0$. Assume $s > 0$. Fix $D = D_\lambda$, $\lambda \in U$, and let P_1, \dots, P_s be its singular points on V . Let V_1, \dots, V_h be the irreducible components of V and let $Q_i \in V_i$ be simple for D ($i = 1, \dots, h$). By (*) there are $E_0, \dots, E_h \in S$ such that E_0 is nonsingular at P_1 and E_i is nontangent to D at Q_i for $i = 1, \dots, h$. Then the generic element E of the linear system generated by E_0, \dots, E_h is nonsingular at P_1 and nontangent to D at Q_1, \dots, Q_h ; hence, by the jacobian criterion of simple points applied to $D \cap E$, it is tangent to D at only finitely many points (other than P_1, \dots, P_s), say R_1, \dots, R_ℓ . Now the generic element of the pencil generated

rated by D and E has at most P_2, \dots, P_s as singular points. On the other hand this pencil corresponds to a line in P^r which intersects U , hence almost all the elements of the pencil are of the form D_μ , $\mu \in U$. But this is a contradiction.

2.2 Corollary : Let the assumptions be as in 2.1 and assume further that $U = X - Z$ be smooth, that Z be the base locus of S and that S separates the tangent vectors on U . Then if $D \in S$ is generic we have : $\text{Sing}(D) \subset Z - V$.

Proof : Apply 2.1 and 1.3.

The assumptions of 2.1 and 2.2 are verified by a large class of linear systems if X is projective. To see this we need first a lemma :

2.3 Lemma : Assume $X \subset P^n$ and let $x \in Z$. Put : $A = O_{X,x}$ and $B = O_{Z,x} = A/a$. Let f_1, \dots, f_s be non zero elements of a . Then for any $t \gg 0$ there are $D_1, \dots, D_t \in S^{(t)}$ whose local equations at x (see 0.7) are f_1, \dots, f_s (as in the previous section $S^{(t)}$ is the linear system cut out on X by the hypersurfaces of degree t which contain Z).

Proof : Let R be the graded ring of X in P^n and let $I, P \subset R$ be the homogeneous ideals of Z and X respectively. Then a is the set of all fractions of the form a/b , where $a, b \in R_t$ (for some t depending on a and b), $a \in a, b \notin P$. Thus there are an integer t_0 and $a_1, \dots, a_s \in a \cap R_{t_0}$, $s \in R_{t_0} - P$, such that $f_i = a_i/s$, and this is also true for all $t \geq t_0$. Then it is easy to see that a_1, \dots, a_s define $D_1, \dots, D_s \in S^{(t)}$ whose local equations at x are f_1, \dots, f_s .

2.4 Corollary : The notations being as in 2.3, assume that A be regular of dimension d and that $\text{emdim } B < d$. Then for all $t \gg 0$ there is $D \in S^{(t)}$ which is smooth at x. Hence the generic element of $S^{(t)}$ is smooth at P.

Proof : Let m, n be the maximal ideals of A, B respectively. Then we have $\dim (m/m^2) > \dim (m/m^2 + a)$, hence $a \notin m^2$. The conclusion follows by 2.3.

2.5 Theorem : Let $X \subset \mathbb{P}^n$ be an irreducible projective variety, let $Z \subset X$ be a closed reduced curve and let V be an open subset of Z . Let $x_1, \dots, x_s \in Z-V$ and let $p = (p_1, \dots, p_s)$ be an s-uple of positive integers. Let $S^{(t)}$ be as in 2.3 ; then put : $S_p^{(t)} = \{D \in S^{(t)} / e_{x_i}(D) \geq p_i\}$, where $e_{x_i}(\cdot)$ denotes multiplicity at x_i . Assume that:

- (i) $\dim X \geq 3$;
- (ii) $V \subset \text{Reg}(X)$;
- (iii) $\text{emdim } O_{Z,x} < \dim X = d$ for all $x \in V$.

Then if $t \gg 0$ and $D \in S_p^{(t)}$ is generic, we have:

- (a) D is nonsingular at each point of V ;
- (b) if moreover $X-Z \subset \text{Reg}(X)$, then $\text{Sing}(D) \subset Z-V$.

Proof : To prove (a) it is enough to show that both (*) and (**) of 2.1 hold, with $I = \text{Sing}(V)$. Assume first that $p = (1, \dots, 1)$, so that $S_p^{(t)} = S^{(t)}$ for all t . Let $x \in V$ and put : $A = O_{X,x}$, $B = O_{Z,x} = A/a$. Let m, n be the maximal ideals of A, B respectively. If $x \in \text{Reg}(V)$, then $a = (f_1, \dots, f_e)$, where $e \geq 2$, and f_1, \dots, f_e are contained in a regular system of parameters of A (use (i) and (ii)). Hence by 2.3 there are $D, E \in S^{(t)}$ (for some t) which are non tangent at x . By the jacobian criterion applied to $D \cap E$ we see that D and E are

non tangent in a neighbourhood of x and, by the compactness of V , we see that $S^{(t)}$ verifies (*) for all $t \gg 0$.

As for (**) it follows immediately from 2.4.

Let now p be general. Fix u such that $S^{(u)}$ verifies (*) and (**).

Let e_i be the minimal multiplicity of the elements of $S^{(u)}$ at x_i (then e_i is the multiplicity of the generic element of $S^{(u)}$ at x_i).

We may assume that $e_i \geq p_i$ if and only if $i \geq e$. Let $v = \sum_{i=1}^e (r_i - e_i)$. Then it is clear that, for all $t \geq u+v$, $S_p^{(t)}$ verifies (*) and (**) (add $t-u$ suitable hyperplanes to suitable elements of $S^{(u)}$). This proves (a). To prove (b) apply 1.9 and the same argument used for (a).

2.6 Corollary : In 2.5 put : $e = \sup\{\text{emdim}(O_{Z,x})/x \in V\}$. Then for generic independent $E_1, \dots, E_n \in S^{(t)}$ (see 1.7), if $t \gg 0$ and $n < d-e$ we have: $E_1 \cap \dots \cap E_n$ is nonsingular at all points of V . If moreover $X-Z \subset \text{Reg}(X)$, then $E_1 \cap \dots \cap E_n$ is nonsingular also in $X-Z$.

Proof : It follows from 2.5 and 1.6, with an argument similar to the one used in 1.6.

2.7 Corollary : Let $Z \subset \mathbb{P}^n$ be a curve and let s be the maximum embedding dimension of the points of Z . Then there is a complete intersection $Y \subset \mathbb{P}^n$, of dimension r and smooth (hence irreducible by [H], III, ex. 5.5) containing Z . In particular:

- (a) if $s < n$, Z is contained in a smooth hypersurface;
- (b) if $s \leq 2$, Z is contained in a smooth irreducible surface.

Remark : When Z is contained in a smooth irreducible surface we say that Z has only plane singularities.

2.8 Proposition : Let $Z \subset P^3$ be a curve of degree d and let $S^{(t)}$ be the linear system of the surfaces of degree t which contain Z . Then:

(a) For $t \geq d$, $S^{(t)}$ is non empty and its generic element is smooth at all points of $\text{Reg}(Z)$;

(b) For $t \geq d+1$ the generic element of $S^{(t)}$ is singular at most at the points of $\text{Sing}(Z)$;

(c) If Z is smooth, then the generic element of $S^{(d+1)}$ is smooth.

In particular Z is contained in a smooth surface of degree $d+1$.

Proof : (a) If F is a cone which projects Z from a point, then $\deg F \leq d$ and hence $S^{(t)}$ is non empty for all $t \geq d$ (add hyperplanes to F). Moreover if $x \in \text{Reg}(Z)$ let $L_1 \neq L_2$ be two lines through x , not tangent to Z , not meeting Z outside x . Pick $y_1 \in L_1, y_1 \neq x$ and let F_1 be the cone projecting Z from y_1 . Then F_1 and F_2 are not tangent at x_j and by adding a suitable number of planes not containing x we see that, for $t \geq d$, $S^{(t)}$ verifies the assumptions of 2.1 and (a) follows.

(b) By 1.3 and 1.8 it is enough to show that $S^{(d)}$ separates the points of $U = P^3 - Z$; by the above remark it is enough to show that the cones which project Z from points of P^3 do the same. Let then $x, y \in U, x \neq y$, and let L be the line joining x and y . Let $z \in Z, z \notin L$, and let L' be the line joining x and z . Then if F is the cone which project Z from x , it is clear that $L' \not\subset F$. Let $w \notin L', w \notin F$, and let G be the cone which projects Z from w . Then it is easy to show that $x \in G, y \notin G$. This proves (b), while (c) is an immediate consequence of (a) and (b).

2.9 Remarks : (i) The results of this section are in part contained in [A-K]. However our methods of proofs seem much simpler.

(ii) Corollary 2.5 and proposition 2.6 are false if $\dim Z > 1$. Indeed if $Z \subset \mathbb{C}P^4$ is a smooth surface which is not a complete intersection, then every hypersurface which contain Z must be singular, for otherwise one could apply the Lefschetz - Grothendieck theorem to show that $Z = F \cap F_1$, where F_1 is another hypersurface. Note however that there are hypersurfaces containing Z and smooth outside Z (by 1.9).

3. Points of high embedding dimension and quasi ordinary singularities.

In this section we study the behaviour of the generic element of a linear system of hypersurfaces containing a reduced curve $Z \subset \mathbb{C}P^n$, at the points of embedding dimension n , specially when these points are "quasi ordinary" singularities (in a sense we are going to introduce).

3.1 Definition : Let $F \subset \mathbb{C}P^n$ be a hypersurface. A point $x \in F$ is said to be ordinary if the projectivized tangent cone $\text{Proj}(\text{gr}(O_{F,x}))$ of F at x is nonsingular.

If x is ordinary we have clearly:

- (i) if x is singular, then the singularity of F at x can be resolved by the blowing up of F centered at x ;
- (ii) if $n \geq 3$, then the projectivized tangent cone at x is irreducible.

3.2 Definition : Let $X \subset \mathbb{C}P^n$ be a closed subvariety and let $x \in X$. We say that x is quasi ordinary (q.o) if there is a hypersurface F containing X such that x is ordinary for F .

If $x \in X$ is q.o. we put :

$\rho(x) = \min\{e_x(F)/F \text{ hypersurface with } x \text{ ordinary, } Y \subset F\}$, where

$e_x(V)$ denotes the multiplicity of the variety V at its point x .

Example : if X is a hypersurface, then x is q.o. if and only if x is ordinary and, if this is the case, $\rho(x) = e_x(X)$.

Observe that the notion of q.o. depends on the embedding $X \hookrightarrow P^n$.

We begin with some more or less obvious remarks.

3.3 Lemma : Let $x \in X \subset P^n$ and put : $R = \text{gr}(O_{P^n, x})$, $R' = \text{gr}(O_{X, x}) = R/J$.

The the following are equivalent :

- (i) x is q.o. and $\rho(x) = \rho$;
- (ii) there is a form $\phi \in J$, of degree ρ , such that $\text{Proj}(R/\phi R)$ is smooth.

Proof : Easy from 2.3.

3.4 Remark : Condition (ii) of 3.3 can be interpreted in the following way : the projectivized tangent cone $\text{Proj}(R')$, naturally embedded in $P^{n-1} = \text{Proj}(R)$ (with its irrelevant component, if any) is a subscheme of a smooth hypersurface. Equivalently : the tangent cone $\text{Spec}(R')$ is a subcone of a cone contained in $k^n = \text{Spec}(R)$ and smooth outside the vertex.

3.5 Corollary : The point $x \in X$ is q.o. with $\rho(x) = 1$ if and only if $\text{emdim } O_{X, x} < n$.

Proof : Apply 3.3 and 2.4.

Now we want to give a sufficient condition for a curve singularity to be q.o.; for this we need a lemma.

3.6 Lemma : Let $n \geq 2$ and let $I \subset \mathbb{P}^n$ be a finite set of cardinality r , not contained in any hyperplane. Let $S^{(t)}$ be the linear system consisting of the hypersurfaces of degree t which contain I . Then if $t \geq n-r+2$ the generic element of $S^{(t)}$ is smooth.

Proof : By 2.2 and 1.9 it is enough to show that if $u = r-n+1$ then

- (i) the base locus of $S^{(u)}$ is I ;
- (ii) for every $P \in I$ there is $D \in S^{(u)}$ which is smooth at P .

Let then $P \notin I$. Since I spans \mathbb{P}^n there are $P_1, \dots, P_n \in I$ which are independent and such that the hyperplane H_0 they span does not contain P . Let P_{n+1}, \dots, P_r be the remaining points of I and let H_i be a hyperplane containing P_{n+i} and not containing P . Then $H_0 \cup \dots \cup H_{n-r}$ is a hypersurface of degree u , which contains I but does not contain P . This proves (i).

The proof of (ii) is quite similar.

3.7 Proposition : Let $Z \subset \mathbb{P}^n$ be a reduced curve and assume $n \geq 3$. Let $x \in Z$ be a point of embedding dimension n and assume that $\text{gr}(O_{Z,x})$ be reduced. Then x is q.o. and $\rho(x) \leq e_x(Z) - n + 2$.

Proof : Let $I = \text{Proj}(\text{gr}(O_{Z,x})) \subset \mathbb{P}^{n-1} = \text{Proj}(\text{gr}(O_{\mathbb{P}^n,x}))$. Then I is finite and also a reduced subscheme of \mathbb{P}^{n-1} containing $r = e_x(Z)$ points, which span \mathbb{P}^{n-1} . Since $\text{gr}(O_{Z,x})$ is reduced it is easy to see that the conclusion follows from 3.6 and 3.3.

3.8 Corollary : Let $Z \subset \mathbb{P}^n$, $n \geq 3$, be a reduced curve and let x be a seminormal point of Z of embedding dimension n . Then x is q.o. and $\rho(x)=2$.

Proof : Recall that x is seminormal if and only if $\text{gr}(O_{Z,x})$ is reduced and $e_x(Z) = \text{emdim}(O_{X,x})$ (see [B] or [D]). Then the conclusion follows from 3.7 and 3.5.

3.9 Lemma : Let S be a linear system in \mathbb{P}^n and assume that S contains a smooth element. Then the generic element of S is smooth.

Proof : Let $S^{(t)}$ be the linear system of the hypersurfaces of degree t and identify $S^{(t)}$ with a projective space \mathbb{P}^r (see section 0). Then using remark 1.13, (ix), we see that the set $\{\lambda \in \mathbb{P}^r / D_\lambda \text{ is smooth}\}$ is open and dense in \mathbb{P}^r . The conclusion follows easily since S is a linear subsystem of some $S^{(t)}$ and corresponds to a linear subspace of \mathbb{P}^r .

3.10 Corollary : Let S be a linear system in \mathbb{P}^n and let x be a base point for S . Assume that x is ordinary with multiplicity e for some element of S and that $e_x(D) \geq e$ for all $D \in S$. Then x is ordinary with multiplicity e for the generic element of S .

Proof : The generic element of S has multiplicity e at x and hence there is a basis D_0, \dots, D_r of S such that $e_x(D_i) = e$ for all i . Let φ_i be the leading form in $\text{gr}(O_{\mathbb{P}^n, x})$ of the local equation of D_i at x (see 0.7).

Then (with the conventions of 0.5) if φ_λ is the leading form of the local equation of D_λ , where $\lambda = (\lambda_0, \dots, \lambda_m)$, we have that for λ generic the degree of φ_λ is e and, if this is the case, $\varphi_\lambda = \sum \lambda_i \varphi_i$. The conclusion follows from 3.9.

3.11 Proposition : Let $n \geq 3$ and let $Z \subset \mathbb{C}P^n$ be a reduced curve. Let V be the set of q.o. points of Z and let $I = \{x_1, \dots, x_s\}$ be the set of points of V with embedding dimension n . Let $e_i = \rho(x_i)$ and let $S_e^{(t)}$ be the linear system of the hypersurfaces of degree t which contain Z and which have multiplicity at least e_i at x_i . Then if $t \gg 0$ and $F \in S_e^{(t)}$ is generic we have:

- (a) $\text{Sing}(F) \subset (Z-V) \cup I$;
- (b) x_i is ordinary for F_i with multiplicity e_i .

Proof : It follows from 3.9 and 2.5.

3.12 Corollary : Let $n \geq 3$ and let $Z \subset \mathbb{C}P^n$ be a reduced curve and let I be the set of points of Z which have embedding dimension n . Assume that $\text{gr}(O_{Z,x})$ be reduced for all $x \in I$. Then the generic hypersurface F of degree $t \gg 0$ containing Z is singular only at the points of I . Moreover each $x \in I$ is ordinary for F and $e_x(F) \leq e_x(Z) - n + 2$.

Proof : Apply 3.10 and 3.7.

We do not know much about q.o. singularities in higher dimension.

We can prove only the following.

3.13 Proposition : Let $x \in X \subset \mathbb{C}P^n$ and assume that $n \geq 2 + \dim O_{X,x}$.

Consider the following conditions :

- (i) x is q.o. ;
 - (ii) if $Z = \text{Proj}(O_{X,x})$ and $z \in Z$, then $\text{emdim}(O_{Z,z}) \leq n - 2$.
- Then (i) \rightarrow (ii). If moreover $\text{gr}(O_{X,x})$ is reduced and $\dim O_{X,x} = 2$, the converse is also true.

Proof : By 3.3 and 3.4 we have easily (i) \rightarrow (ii). Conversely if $\dim O_{X,x} = 2$ we have that Z is a reduced curve in $P^{n-1} = \text{Proj}(\text{gr}(O_{P^n,x}))$. Hence if (ii) holds Z is contained in a smooth hypersurface by 2.5. Since $\text{gr}(O_{X,x})$ is reduced, it is easy to deduce (i), using 3.3.

3.14 Remarks : (i) Consider the curve C in P^3 given by the ideal $I = (x_0^2 x_3 - x_1^3, x_0^2 x_3 - x_2^3)$ and let $x = (0,0,0,1)$. Then x is singular for C and moreover the leading ideal of I_x in $\text{gr}(O_{P^n,x}) = k[T_0, T_1, T_2]$ is $a = (T_0^2, T_1^2)$ (e.g. apply [v-v], prop. 1.2). Now it is clear that if $\varphi \in a$ is any form, then $\text{Proj}(k[T_0, T_1, T_2]/(\varphi))$ must be singular at $(0,0,1)$. Hence x is not a q.o. singularity for C . (ii) In 3.6 we have shown that any set I of r points in P^n which spans P^n is contained in a smooth hypersurface of degree $t = r - n + 2$. Given n and r , the above number is clearly not the minimum one that has the property: we do not know which is the minimum one. Note its relation with the character $\rho(x)$ of a q.o. singularity (see 3.7).

(iii) The implication (ii) \rightarrow (i) of 3.13 is false if $\dim O_{X,x} > 2$. Indeed let $Z \subset P^n$ be a smooth irreducible surface which is not a complete intersection and let $V \subset k^5$ be the corresponding affine cone. Embed k^5 in P^5 and let X be the closure of V in P^5 . Let $x \in X$ be the vertex of the cone V . Then $\dim O_{X,x} = 3$, $\text{gr}(O_{X,x}) =$ ring of coordinates of $V \subset k^5$ and hence $\text{Proj}(\text{gr}(O_{X,x})) = Z$. So the given embedding of Z in P^4 coincides with the embedding $Z \subset \text{Proj}(\text{gr}(O_{P^5,x})) = P^4$. Hence Z verifies (ii) of 3.13 and moreover $\text{gr}(O_{X,x})$ is reduced. However Z is not contained in any smooth hypersurface (see 2.8 (ii)); hence x is not q.o.

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Authors' address : Istituto matematico - Politecnico di Torino (Italy)

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